Standard Cosmology on a Self-Tuning Domain Wall

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Abstract

We investigate the cosmology of (4+1)-dimensional gravity coupled to a scalar field and a *bulk* anisotropic fluid within the context of the single-brane Randall-Sundrum scenario. Assuming a separable metric, a static fifth radius and the scalar to depend only on the fifth direction, we find that the warp factor is given as in the papers of Kachru, Schulz and Silverstein [hep-th/0001206, hep-th/0002121] and that the cosmology on a self-tuning brane is standard. In particular, for a radiation-dominated brane the pressure in the fifth direction vanishes.

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1 Introduction

Theories with extra dimensions where our four-dimensional world is a hypersurface (three-brane) embedded in a higher-dimensional spacetime and at which gravity is localised have been the subject of intense scrutiny since the work of Randall and Sundrum [1]. The main motivation for such models comes from string theory where they are reminiscent of the Hořava-Witten solution [2] for the field theory limit of the strongly-coupled $E_8 \times E_8$ heterotic string. The Randall–Sundrum (RS) scenario may be modelled [3,4] by coupling gravity to a scalar field and mapping to an equivalent supersymmetric quantum mechanics problem. A static metric is obtained with a warp factor determined by the superpotential. A generalisation to non-static metrics was considered by Binétruy, Deffayet and Langlois (BDL) who modelled brane matter as a perfect fluid delta-function source in the five-dimensional Einstein equations [5]. However, this resulted in non-standard cosmology in that the square of the Hubble parameter on the brane was not proportional to the density of the fluid. Other cosmological aspects of "brane-worlds" have been considered in [6].

In this letter, we investigate RS-type single brane cosmological solutions of five-dimensional gravity coupled to a scalar field which we assume to depend only on the fifth dimension. We further assume that the fifth dimension is static and infinite in extent. We also include a *bulk* anisotropic fluid with energy-momentum tensor $\hat{T}^A_B(\rho) = \mathrm{diag}(-\rho, p, p, p, P)$ and equations of state $P = \tilde{\omega}\rho$, $p = \omega\rho$. Assuming a separable metric, we find that the warp factor is given as in the papers of Kachru, Schulz and Silverstein (KSS) [7, 8]. We also find that the cosmology on a self-tuning brane is standard but that the pressure in the fifth direction is constrained by the relation $\tilde{\omega} = \frac{3\omega-1}{2}$. In particular, we find that the pressure in the fifth direction vanishes for a radiation-dominated brane with $\omega = 1/3$.

2 The Model

We consider a single, thin brane at $r = 0$, as in KSS [7]. The action for the gravity and scalar part of the model is:

$$\begin{aligned} S &= S_{gravity} + S_{bulk} + S_{brane} \ , \\ S_{gravity} &= \frac{1}{2\hat{\kappa}_5^2} \int d^4x dr \sqrt{-\hat{g}}\, \hat{R} \ , \\ S_{bulk} &= \int d^4x dr \sqrt{-\hat{g}} \left(-\frac{4}{3}\, \hat{g}^{AB} \partial_A \Phi \partial_B \Phi - U(\Phi) \right) \ , \end{aligned}$$

$$S_{brane} = \int d^4x \sqrt{-g^{(4)}} \, (-V(\Phi)) \ , \tag{1}$$

where $\hat{g}_{AB}$ is the five-dimensional metric, $g^{(4)}_{ij}$ is the induced metric on the brane and the tension of the brane is parametrised by $V(\Phi)$.

We assume a separable metric with flat spatial three-sections on the brane:

$$\begin{aligned} ds^2 &= \hat{g}_{AB} dy^A dy^B \\ &= e^{2A(r)} \left(-dt^2 + g(t)\delta_{ab} dx^a dx^b \right) + dr^2 \ . \end{aligned} \tag{2}$$

This is a natural generalisation of the $4d$ flat Robertson-Walker metric to a RS context and is a special case of the BDL ansatz (see [5]) with $n(t,r) = e^{A(r)}$, $a(t,r) = e^{A(r)} g^{1/2}(t)$, $b(t,r) = 1$ in conventional notation.

We shall also make the ansatz that both the potentials $U(\Phi)$ and $V(\Phi)$ are of Liouville type (see, for instance, [9]):

$$\begin{aligned} U(\Phi) &= U_0 e^{\alpha\Phi} \ , \\ V(\Phi) &= V_0 e^{\beta\Phi} \ , \end{aligned} \tag{3}$$

where U_0 and V_0 are constants.

The stress-tensor for the scalar is

$$\hat{T}^A_B(\Phi) = \check{T}^A_B + \tilde{T}^A_B \ , \tag{4}$$

where

$$\check{T}^A_B = \frac{8}{3} \partial^A \Phi \partial_B \Phi - \delta^A_B \Big(\frac{4}{3} \partial^C \Phi \partial_C \Phi + U(\Phi) \Big) \ , \tag{5}$$

and

$$\tilde{T}^A_B = - \frac{\sqrt{-g^{(4)}}}{\sqrt{-\hat{g}}} \, V(\Phi) \, \delta(r) \, g^{(4)}_{ij} \, \delta^{iA} \, \delta^j_B \ , \tag{6}$$

where there is no sum over the indices i and j. We shall assume that Φ depends only on r.

The bulk fluid has the stress-tensor [10]:

$$\hat{T}^A_B(\rho) = \mathrm{diag}\,(-\rho, p, p, p, P) \tag{7}$$

in the comoving coordinates y^A. ρ is the density and p and P are the pressures in the three spatial directions on the brane and in fifth dimension, respectively. The anisotropy can be considered as a result of the mixing of two interacting perfect fluids [11].

3 The Solutions

We now proceed to solve Einstein's equations $\hat{G}^A_B = \hat{\kappa}_5^2(\hat{T}^A_B(\Phi)+\hat{T}^A_B(\rho))$ given the above ansatze.

If we take a linear combination of the 00- and 11-components of Einstein's equations then the following equation results:

$$\frac{\dot{g}^2}{g^2} - \frac{\ddot{g}}{g} - \hat{\kappa}_5^2 e^{2A}(\rho + p) = 0 \ . \tag{8}$$

Therefore, we see that ρ and p must be of the form

$$\begin{aligned} \rho(t,r) &= e^{-2A(r)}\,(\tilde{\rho}(t) + F(t,r)) \ , && (9) \\ p(t,r) &= e^{-2A(r)}\,(\tilde{p}(t) - F(t,r)) \ , && (10) \end{aligned}$$

for arbitrary $F(t,r)$. However, it is normal to assume the equation of state $p = \omega\rho$, where ω is constant in the range $-1 \leq \omega \leq 1$. In the generic case $\omega \neq -1$ this implies that F should be zero. We shall assume this also to be so in the special case $\omega = -1$. Furthermore, we shall also assume $P = \tilde{\omega}\rho$. Equation (8) then reduces to

$$\frac{\dot{g}^2}{g^2} - \frac{\ddot{g}}{g} - \hat{\kappa}_5^2\,(1+\omega)\,\tilde{\rho} = 0 \ . \tag{11}$$

Given $F = 0$, the 00-component of Einstein's equations separates into

$$\frac{3}{4}\,\frac{\dot{g}^2}{g^2} - \hat{\kappa}_5^2\,\tilde{\rho} = C \ , \tag{12}$$

$$(6A'^2 + 3A'') + \frac{4\,\hat{\kappa}_5^2}{3}\,\Phi'^2 + \hat{\kappa}_5^2\,U + \hat{\kappa}_5^2\,V\,\delta(r) = C\,e^{-2A} \ , \tag{13}$$

where C is the separation constant.

The rr-equation also splits in two:

$$\frac{3}{2}\,\frac{\ddot{g}}{g} + \hat{\kappa}_5^2\,\tilde{\omega}\,\tilde{\rho} = D \, , \tag{14}$$

$$6A'^2 - \frac{4\,\hat{\kappa}_5^2}{3}\,\Phi'^2 + \hat{\kappa}_5^2\,U = D\,e^{-2A} \ , \tag{15}$$

where D is another separation constant.

Equation (15) allows us to recast (13) in the form

$$3A'' + \frac{8\,\hat{\kappa}_5^2}{3}\Phi'^2 + (D-C)\,e^{-2A} + \hat{\kappa}_5^2\,V\,\delta(r) = 0 \ . \tag{16}$$

We shall see below that in fact $D = 2\,C$.

In addition, the equation of motion for the scalar field

$$\frac{8}{3}\,\hat{\nabla}^2\Phi - \frac{\partial U(\Phi)}{\partial\Phi} - \frac{\sqrt{-g^{(4)}}}{\sqrt{-\hat{g}}}\,\frac{\partial V(\Phi)}{\partial\Phi}\,\delta(r) = 0 \ , \tag{17}$$

results in the equation

$$\frac{8}{3}\Phi'' + \frac{32}{3}A'\Phi' - \alpha\,U - \beta\,V\delta(r) \ \ = \ \ 0 \ , \tag{18}$$

Note that the scalar field equation of motion implies that $\hat{\nabla}^A\check{T}_{AB} = 0$ (and, conversely, off the brane only). This, in turn, implies that the fluid equations of motion $\hat{\nabla}^A\hat{T}_{AB}(\rho) = 0$ are automatically satisfied.

The Warp Factor

Equations (15), (16) (with $D = 2C$) and (18) have been extensively studied in [7,8,12–14]. The self-tuning domain wall (solution (I) of [7]) is given by

$$U = C = 0 \ , \ \ \beta \neq \pm\frac{1}{a} \ , \tag{19}$$

$$\begin{aligned} \Phi(r) &= a\,\epsilon\log(d - cr) \ , & (20)\\ A(r) &= \frac{1}{4}\log(d - cr) - e \ , & (21) \end{aligned}$$

where $a = 3/(4\sqrt{2}\,\hat{\kappa}_5)$ and ϵ is a sign that takes opposite values either side of the brane at $r = 0$. The parameters c, d and e are constants of integration that can also differ either side of the brane. For (20) to make sense, we require $d > 0$. The continuity of Φ and A across the brane requires

$$d_+d_- = 1 \ , \tag{22}$$

$$e_+ = \frac{1}{4}\log d_+ \ , \ \ e_- = \frac{1}{4}\log d_- \ , \tag{23}$$

where we have chosen the convention $A(0) = 0$ and denoted constants defined on the right (left) side of the the brane with a $+(-)$ subscript. The jump condtions implied by (16) and (18) result in the relations

$$\begin{aligned} c_+ &= -\frac{2}{3}\hat{\kappa}_5^2\,d_+\,(a\beta\epsilon_+ - 1)\,V_0\,e^{a\beta\epsilon_+\log d_+} \ , \\ c_- &= -\frac{2}{3}\hat{\kappa}_5^2\,d_-\,(a\beta\epsilon_+ + 1)\,V_0\,e^{a\beta\epsilon_+\log d_+} \ . & (24) \end{aligned}$$

The solution is self-tuning because given d_+, $\epsilon_+ = \pm 1$ and $\beta \neq \pm 1/a$, there is a Poincaré-invariant four-dimensional domain wall for any value of the brane tension V_0; V_0 does not need to be fine-tuned to find a solution.

Other warp factors are possible both when $C = 0$ and when $C \neq 0$. Solution (II) of [7] with $U = 0$ and solution (III) of the same reference with $U \neq 0$ are examples of the former case. The solution presented in [8] with $U = 0$ provides an example the latter.

The Cosmology

Adding equations (12) and (14) gives:

$$\dot{g}^2 + 2g\ddot{g} + \frac{4}{3}\,\hat{\kappa}_5^2\, g^2(\tilde{\omega} - 1)\,\tilde{\rho} = \frac{4}{3}(D + C)\,g^2 \ . \tag{25}$$

On the other hand, using (12) in (11) we obtain:

$$\dot{g}^2 + 2g\ddot{g} + 2\,\hat{\kappa}_5^2 g^2\,(\omega - 1)\,\tilde{\rho} = 4C\,g^2 \ . \tag{26}$$

Since $\tilde{\rho}$ is generically a function of t (rather than a constant) the above two equations imply the relations

$$D = 2\,C \tag{27}$$

$$\omega = \frac{1}{3}\,(1 + 2\tilde{\omega}) \ . \tag{28}$$

Relation (28) previously appeared in [15]. In particular, it implies that an isotropic (perfect) fluid ($P = p$) is stiff, that is, $\omega = \tilde{\omega} = 1$. The attribute "stiff" refers to the fact that the velocity of sound in the fluid is equal to the velocity of light. It should be noted that the case of a bulk cosmological constant ($\omega = \tilde{\omega} = -1$) is not covered here; however, it corresponds to the choice $U(\Phi) = \mathit{constant}$ instead.

Using (12), equation (25) may be expressed alternatively as

$$\tilde{\omega}\,\dot{g}^2 + 2g\,\ddot{g} = \frac{4}{3}(\tilde{\omega} + 2)C\,g^2 \ , \tag{29}$$

with the solutions*

$$g \sim \begin{cases} \sinh^{2q}\!\left(\sqrt{\frac{C}{3q^2}}\,t\right) & C > 0 \ , \\ t^{2q} & C = 0 \ , \\ \sin^{2q}\!\left(\sqrt{\frac{|C|}{3q^2}}\,t\right) & C < 0 \ , \end{cases} \tag{30}$$

*These solutions, in the particular case of an isotropic fluid, appeared in a different setting in [16].

where $q = 1/(2+\tilde{\omega}) = 2/(3(1+\omega)) = q_{standard}$.

From (12) we see that the density $\tilde{\rho}$ (which is actually the density of the fluid on the brane since we have defined $A(0) = 0$) is positive:

$$\tilde{\rho}(t) = \begin{cases} \hat{\kappa}_5^{-2}\, C \, \sinh^{-2}(\sqrt{\frac{C}{3q^2}}\, t) & C > 0 \ , \\ \frac{3\,\hat{\kappa}_5^{-2}\, q^2}{t^2} & C = 0 \ , \\ \hat{\kappa}_5^{-2}\, |C| \, \sin^{-2}(\sqrt{\frac{|C|}{3q^2}}\, t) & C < 0 \ . \end{cases} \tag{31}$$

When $C \geq 0$, equation (29) also allows the de-Sitter solutions $g = e^{\pm 2\sqrt{C/3}\, t}$. These solutions have vanishing density $\tilde{\rho}$ and were discussed in [3, 7, 8, 17]. For the case $C = 0$, we obtain conventional cosmology $H = \dot{a}/a \propto \sqrt{\tilde{\rho}}$ on the brane with evolution at the standard rate.

Of particular note is the case of radiation-dominated fluid on the brane ($\omega = 1/3$). From (28) we see that the pressure in the fifth direction vanishes and the stress tensor is then:

$$\hat{T}^A_B(\rho) = e^{-2A(r)}\, \tilde{\rho}(t)\, \mathrm{diag}(-1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0) \ , \tag{32}$$

with $q_{standard} = 1/2$.

4 Summary

The self-tuning domain wall, with warp factor given by (21), has vanishing separation constant C and therefore expands according to the power law (30) at the standard rate and exhibits conventional cosmology when coupled to a *bulk* anisotropic fluid. The pressure of the fluid in the fifth direction, P, is related to the isotropic pressure on the brane, p, via equation (28) and vanishes for a radiation-dominated brane.

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